UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH2401**

MODULE NAME : **Mathematical Methods 3**

DATE : **08-May-07**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2006/07-MATH2401A-001-EXAM-167

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. The *steady* temperature distribution $\theta(x, y)$ of a square sheet of metal of unit area, which has its upper and lower surfaces insulated, satisfies Laplace's equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0, \qquad 0 \le x \le 1, \;\; 0 \le y \le 1.$$

(a) Derive the general solution for $\theta(x, y)$ in variables separable form.

(b) Find the temperature everywhere in the sheet if the temperature is held constant ($\theta = 0$) on three edges ($y = 0$, $y = 1$ and $x = 0$) and the fourth edge at $x = 1$ is heated so that

$$\theta(1, y) = 1.$$

(c) Using arguments involving symmetry and linearity, or otherwise, write down the steady solution if the edge at $y = 0$ is also heated so that

$$\theta(x, 0) = 1,$$

with the temperatures on the other three edges as in (b).

2. Two identical strings of length L and density ρ, are knotted together at $x = 0$, with the knot having mass M. The other ends are attached to points $2L$ apart so that each string is under tension T. Small transverse disturbances in each string have amplitude $z_1(x, t)$, $(-L \le x \le 0)$ and $z_2(x, t)$, $(0 \le x \le L)$ respectively, and one of the boundary conditions at $x = 0$ can be shown to be

$$M\frac{\partial^2 z_1}{\partial t^2} = T\left(\frac{\partial z_2}{\partial x} - \frac{\partial z_1}{\partial x}\right).$$

(a) Write down the partial differential equation that describes the time evolution of the displacements z_1 and z_2 in the strings, defining any constants that you introduce, as well as the remaining boundary conditions at $x = -L, 0, L$.

(b) Show, by writing the *time dependent* part of the disturbance in complex form, or otherwise, that the normal modes of vibration of the string that are symmetric about $x = 0$ have period $2\pi/\omega$, where ω satisfies

$$\frac{\omega L}{c} \tan\left(\frac{\omega L}{c}\right) = \frac{2\rho L}{M},$$

and c is the wave speed in the string. What happens in the limit $M \to 0$? What values does ω take for the normal modes that are anti-symmetric about $x = 0$?

3. A heavy chain of length L and density per unit length ρ hangs under gravity between two fixed points on the same horizontal level, which are a distance a apart. The potential energy V of the chain may be written

$$V = -\rho g \int_0^a y\left[1 + (y')^2\right]^{1/2} dx,$$

if coordinates (x, y) are chosen with the x-axis horizontal and the origin at one end of the chain.

Write down an integral expression for the length L of the chain.

When the chain hangs in stable equilibrium, the potential energy V is a minimum. Deduce that $y(x)$ then satisfies

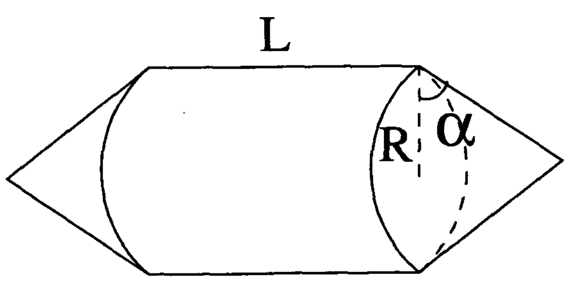
$$\left(\frac{dy}{dx}\right)^2 = k^2(y+h)^2 - 1,$$

where h and k are constants. Hence show that the equilibrium shape of the chain is a **catenary** with equation

$$y = -h + k^{-1}\cosh\{kx + \alpha\},$$

where $\alpha = -ka/2$. Write down, but do not attempt to solve, the equations which determine the constants k and h.

4. A machine component, as illustrated in the diagram, consists of a open circular cylinder of radius R and length L, attached at each end to a cone of base angle α, which also has base radius R. The component is constrained to have fixed surface area A. Find, in terms of A, the dimensions R, L and α of the component that maximize its volume, and find the maximum volume V.



[Recall that the volume V_c and surface area A_c of a (baseless) cone are given by

$$V_c = \frac{1}{3}\pi R^3 \tan\alpha, \quad A_c = \pi R^2 \sec\alpha,$$

respectively.]

5. Find the general solutions of the following partial differential equations

(a)
$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{1}{x^2 - y^2},$$

(b)
$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = \sin(x + 3y),$$

(c)
$$xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = x^2 - y^2.$$

In the case of (c) you may leave the solution in implicit form.

6. For each of the following partial differential equations, find the solution $z(x,t)$, valid in the domain $-\infty < x < \infty$, $t \geq 0$, that satisfies the given initial conditions at $t = 0$. To illustrate your solutions, sketch on an (x,t) diagram those regions where the solution takes a different form.

(a)
$$2x\frac{\partial z}{\partial t} + \frac{1}{(t+1)^2}\frac{\partial z}{\partial x} = 0, \qquad z(x,0) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$$

(b)
$$(\dagger) \quad \frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}, \qquad z(x,0) = \begin{cases} 1, & x < 0 \\ 0, & x \geq 0 \end{cases}, \qquad \frac{\partial z}{\partial t}(x,0) = 0.$$

[In (b), you may assume without proof that D'Alembert's solution of the one-dimensional wave equation (†), given by

$$z(x,t) = \frac{1}{2}[F(x+t) + F(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} G(\zeta)\, d\zeta$$

for initial conditions

$$z(x,0) = F(x), \qquad \frac{\partial z}{\partial t}(x,0) = G(x),$$

holds in the domain $-\infty < x < \infty$, $t \geq 0$.]

END OF PAPER